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Some properties of Schur-constant survival models and their copulas

Roger B. Nelsen
Lewis & Clark College

Abstract: We study continuous, nonnegative random variables with a Schur-constant joint survival function. We show that these distributions are characterized by having an Archimedean survival copula, determine the distributions of certain functions of the random variables, and examine dependence properties and correlation coefficients for random variables with Schur-constant survival functions.

Keywords: Archimedean copula, Pareto distribution, Schur-constant function, survival function, Weibull distribution.

1 Introduction

Let $\mathbf{X} = (X_1,\cdots,X_n)$ be a vector of n continuous, non-negative random variables (which we call *lifetimes*) with a *Schur-constant* joint survival function, i.e., for any $\mathbf{x} = (x_1,\cdots,x_n)$ in $[0,\infty)^n$,

$$P(\mathbf{X} > \mathbf{x}) = P\left(X_1 > x_1,\cdots,X_n > x_n\right) = S(x_1 + \cdots + x_n) \tag{1.1}$$

for an appropriate function S (note that an inequality between vectors is componentwise). Schur-constant survival models and some of their basic properties were studied in Barlow and Mendel (1993); Caramellino and Spizzichino (1994, 1996); and Spizzichino (2001). It is immediate that if $\mathbf{X}$ has a Schur-constant joint survival function, then the components of $\mathbf{X}$ are exchangeable, and all the lower dimensional marginal joint survival functions are also Schur-constant. Lifetimes with Schur-constant joint survival functions are of interest in reliability theory because of their property of *indifference relative to aging*, e.g. Barlow and Mendel (1993), or the *no-aging* property, e.g. Caramellino and Spizzichino (1994). Note that for any $i \neq j$ in $\{1,2,\cdots,n\}$, any $\mathbf{x}$ in $[0,\infty)^n$, and any $t \geq 0$,

$$P\left(X_i - x_i > t \middle| \mathbf{X} > \mathbf{x}\right) = \frac{S(x_1 + \cdots + x_n + t)}{S(x_1 + \cdots + x_n)} = P\left(X_j - x_j > t \middle| \mathbf{X} > \mathbf{x}\right),$$

that is, the residual lifetimes $\left(X_i - x_i\right)$ and $\left(X_j - x_j\right)$ of two components of different ages x_i and x_j, respectively, have the same conditional distribution.

Some basic properties of the function S in (1.1) are readily obtained. Setting all but one component of $\mathbf{x}$ to 0 yields $P\left(X_i > x_i\right) = S(x_i)$, thus S is a univariate survival function, i.e., S is continuous and nonincreasing with $S(0) = 1$ and $S(\infty) = 0$. Furthermore, $S(x_1 + \cdots + x_n)$ must be a multivariate joint survival function. To examine what this implies, let $\mathbf{x}$ and $\mathbf{y}$ be any two points in $[0,\infty)^n$ such that $\mathbf{x} \le \mathbf{y}$, and let $[\mathbf{x},\mathbf{y}]$ denote the *n-box* $[x_1,y_1] \times [x_2,y_2] \times \cdots \times [x_n,y_n]$. The *vertices* of the n-box $[\mathbf{x},\mathbf{y}]$ are the points $\mathbf{z} = (z_1,\cdots,z_n)$ where each z_k is equal to either x_k or y_k. Since the probability mass assigned to an n-box $[\mathbf{x},\mathbf{y}]$ by S must be non-negative, the inclusion-exclusion principle dictates that

$$\sum \operatorname{sgn}(\mathbf{z}) S(z_1 + \cdots + z_n) \ge 0, \tag{1.2}$$

where the sum is over the vertices $\mathbf{z}$ of $[\mathbf{x},\mathbf{y}]$ and $\operatorname{sgn}(\mathbf{z}) = 1$ if $z_k = x_k$ for an even number of ks, and $\operatorname{sgn}(\mathbf{z}) = -1$ if $z_k = x_k$ for an odd number of ks.

Many authors who have studied Schur-constant survival models have done so in the context of absolutely continuous survival functions, see e.g. Barlow and Mendel (1993); Caramellino and Spizzichino (1994, 1996); Spizzichino (2001), however, we will not make this assumption. We first consider the bivariate case in the next section, in order to study relationships (aging, dependence, correlation, etc.) for pairs of components of $\mathbf{X}$ in Section 3. In Section 4 we study the case the case of arbitrary n. In each case we show that $\mathbf{X}$ has an Archimedean survival copula whose generator is the inverse of S.

We conclude this section by examining independent random variables with a Schur-constant survival function.

Theorem 1 *Let* $\mathbf{X}$ *be a vector of* $n \ge 2$ *lifetimes with a Schur-constant survival function given by* (1.1). *Then the components of* $\mathbf{X}$ *are independent if and only if they are exponentially distributed.*

Proof. Assume the components of $\mathbf{X}$ are independent, and let $\mathbf{x}$ be any element of $[0,\infty)^n$. Partition the components of $\mathbf{x}$ into two nonempty sets, say $\{x_{i_1},\cdots,x_{i_m}\}$ and $\{x_{i_{m+1}},\cdots,x_{i_n}\}$ for some m between 1 and n–1, inclusive. Then $S(u+v) = S(u)S(v)$ where $u = x_{i_1} + \cdots + x_{i_m}$ and $v = x_{i_{m+1}} + \cdots + x_{i_n}$. Thus S satisfies Cauchy's equation on $[0,\infty)$, and the solution is given by $S(t) = e^{-\lambda t}$, with $\lambda > 0$ since S is nonincreasing. Hence each component of $\mathbf{X}$ is exponentially distributed with parameter λ. The converse is trivial. ❑

As a consequence, the "no-aging" property of random vectors with a Schur-constant survival function is a generalization of the "lack of memory" property for exponential random variables.

2 Bivariate Schur-constant survival models

Throughout this section we let X and Y denote any two components of $\mathbf{X}$, or equivalently, we let $n = 2$ and set $\mathbf{X} = (X,Y)$ and $\mathbf{x} = (x,y)$. As before, S denotes the common survival function for X and Y. We first find a necessary and sufficient condition for S to satisfy (1.2) for $n = 2$, i.e., for $S(x + y)$ to be a bivariate survival function.

Lemma 1 *Let* $S:[0,\infty]\to[0,1]$ *be a continuous survival function* (*i.e.*, S *is continuous, nonincreasing,* $S(0) = 1$ *and* $S(\infty) = 0$). *Then* $S(x + y)$ *is a bivariate survival function if and only if* S *is convex.*

Proof. A function $K:[0,\infty)^2 \to [0,1]$ is a bivariate survival function if and only if its margins $K(x,0)$ and $K(0,y)$ are univariate survival functions and K is 2-*increasing*, i.e., for any x_1, x_2, y_1, y_2 in $[0,\infty]$ with $x_1 < x_2$, $y_1 < y_2$,

$$\Delta_K(x_1,x_2;y_1,y_2) = K(x_1,y_1) - K(x_1,y_2) - K(x_2,y_1) + K(x_2,y_2) \geq 0. \quad (2.1)$$

Let $K(x,y) = S(x + y)$ with S convex. Since $K(x,0) = S(x)$ and $K(0,y) = S(y)$, the margins of K are survival functions. To show that K is 2-increasing, we need to show that

$$S(x_1 + y_2) + S(x_2 + y_1) \leq S(x_1 + y_1) + S(x_2 + y_2),$$

i.e., that S satisfies (1.2). Since this inequality holds if either x_2 or y_2 is infinite, we assume $x_2 < \infty$ and $y_2 < \infty$ and let

$$t = \frac{x_2 - x_1}{(x_2 - x_1) + (y_2 - y_1)} \text{ and } 1 - t = \frac{y_2 - y_1}{(x_2 - x_1) + (y_2 - y_1)},$$

so that $x_1 + y_2 = t(x_1 + y_1) + (1 - t)(x_2 + y_2)$. The convexity of S implies that $S(x_1 + y_2) \leq tS(x_1 + y_1) + (1 - t)S(x_2 + y_2)$. Similarly $x_2 + y_1 = (1 - t)(x_1 + y_1) + t(x_2 + y_2)$, so that $S(x_2 + y_1) \leq (1 - t)S(x_1 + y_1) + tS(x_2 + y_2)$; hence $S(x_1 + y_2) + S(x_2 + y_1) \leq S(x_1 + y_1) + S(x_2 + y_2)$, as required.

Now assume that K is a bivariate survival function, and let $0 \leq a < b$. Then

$$0 \leq \Delta_K(a/2,b/2;a/2,b/2) = S(a) - 2S\big((a+b)/2\big) + S(b),$$

and hence $S\big((a+b)/2\big) \leq \lfloor S(a) + S(b) \rfloor/2$, that is, S is midconvex. Since S is continuous, it follows that S is convex. ❑

In order to describe the dependence structure for random variables with a Schur-constant joint survival function, we need the notion of a *copula.* A (two-dimensional) copula is a function $C:[0,1]^2 \to [0,1]$ such that $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(t,1) = t$ for all t in $[0,1]$, and which is 2-increasing in the sense of (2.1), i.e., $\Delta_C(u_1,u_2;v_1,v_2) \geq 0$ for all u_1, u_2, v_1, v_2 in $[0,1]$ with $u_1 < u_2$, $v_1 < v_2$. Copulas are important in statistical modeling since they join or "couple" joint distributions to their one-dimensional margins, and also couple joint survival functions to their one-dimensional marginal survival functions (in this case, the term "survival copula" is often employed). In our case $P(X > x, Y > y) = C(S(x),S(y))$ for some copula C. See Nelsen (1999) for further details.

An important family of copulas are the *Archimedean* copulas, which have the form $C(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v))$, where φ (the *generator* of C) is a continuous strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$, and where $\varphi^{[-1]}$ denotes the *pseudo-inverse* of φ given by $\varphi^{[-1]}(t) = \varphi^{-1}(t)$ for $t \in [0,\varphi(0)]$ and 0 for $t > \varphi(0)$. Note that φ is the (ordinary) inverse of $\varphi^{[-1]}$ on $[0,\varphi(0)]$.

Theorem 2 *Let X and Y be lifetimes with a Schur-constant survival function* $\Pr(X > x, Y > y) = S(x+y)$ *for all* $x,y \geq 0$. *Then X and Y possess an Archimedean survival copula whose generator is the inverse of S.*

Proof. Let $\varphi_0 = \inf\{x | S(x) = 0\}$ (if $S(x) > 0$ for all $x \geq 0$, we set $\varphi_0 = \infty$). Since S is nonincreasing and convex on $[0,\infty]$ with $S(0) = 1$ and $S(\infty) = 0$, it must be decreasing on $[0,\varphi_0]$. Hence there exists a function $\varphi : [0,1] \to [0,\infty]$, decreasing and convex with $\varphi(0) = \varphi_0$ and $\varphi(1) = 0$, such that $S(\varphi(t)) = t$ for all t in $[0,1]$, and $\varphi(S(x)) = \min(x,\varphi_0)$ for all $x \geq 0$. Thus φ is the inverse of S (on $[0,\varphi_0]$) while S is the pseudo-inverse $\varphi^{[-1]}$ of φ. If either x or y is larger than φ_0, then $S(x+y) = 0$, so that

$$P(X > x, Y > y) = S(x+y) = S(\min(x,\varphi_0) + \min(y,\varphi_0)) = S(\varphi(S(x)) + \varphi(S(y))).$$

Hence $P(X > x, Y > y) = C(S(x), S(y))$ where C is the Archimedean copula generated by φ, given by $C(u,v) = S(\varphi(u)+\varphi(v))$, u,v in $[0,1]$. ❑

A converse of Theorem 2 also holds.

Theorem 3 *If U and V are uniform* (0,1) *random variables whose joint distribution function is an Archimedean copula with generator* φ, *then the random variables* $X = \varphi(U)$ *and* $Y = \varphi(V)$ *have a Schur-constant joint survival function.*

Proof. Let φ be a decreasing convex function from $[0,1]$ to $[0,\infty]$ with $\varphi(1) = 0$, and let U and V be uniform (0,1) random variables whose joint distribution function is the Archimedean copula $C(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v))$ generated by φ. Let $X = \varphi(U)$ and $Y = \varphi(V)$. Since $\varphi^{[-1]}(\varphi(t)) = t$ for all t in $[0,1]$, the survival function of X (and similarly of Y) is given by

$$P(X > x) = P(\varphi(U) > x) = P\left(U < \varphi^{[-1]}(x)\right) = \varphi^{[-1]}(x)$$

for all $x \geq 0$. Hence $\varphi^{[-1]}$ is the survival function of X and of Y. So for all $x,y \geq 0$,

$$\begin{aligned} P(X > x, Y > y) &= P(\varphi(U) > x, \varphi(V) > y) = P\left(U < \varphi^{[-1]}(x), V < \varphi^{[-1]}(y)\right), \\ &= \varphi^{[-1]}\left(\varphi\left(\varphi^{[-1]}(x)\right) + \varphi\left(\varphi^{[-1]}(y)\right)\right), \\ &= \varphi^{[-1]}(\min(x,\varphi(0)) + \min(y,\varphi(0))) = \varphi^{[-1]}(x+y), \end{aligned}$$

thus X and Y have a Schur-constant joint survival function. ❑

Remark: For any positive constant c, φ and $c\varphi$ generate the same Archimedean copula, and hence there is a one-to-one correspondence between bivariate Schur-constant models (modulo a scale parameter) and Archimedean copulas [note that if $\varphi(t)$ is the inverse of $S(x)$, then $c\varphi(t)$ is the inverse of $S(x/c)$].

In the next theorem we present properties of the lifetime proportions $X/(X+Y)$ and $Y/(X+Y)$, the lifetime ratios Y/X and X/Y, the order statistics max(X,Y) and min(X,Y), the absolute difference $|X-Y|$ of lifetimes, and the regression functions $E(Y|x)$ and $E(X|y)$. The proof is straightforward and hence omitted (except to note that the convexity of S guarantees the existence and continuity of S' almost everywhere in $(0,\infty)$).

Theorem 4 *Let X and Y be lifetimes with a Schur-constant survival function* $P(X > x, Y > y) = S(x+y)$ *for all* $x,y \geq 0$. *Then*

(a) *the survival function of the total lifetime* $X+Y$ *is* $P(X+Y>t) = S(t) - tS'(t)$;

(b) *the proportion* $X/(X+Y)$ *is uniformly distributed on* (0,1) *and is independent of the total lifetime* $X+Y$ (*and similarly for* $Y/(X+Y)$);

(c) *the ratio* Y/X *has a Pareto distribution with shape parameter* 1 (*i.e.*, $P(Y/X > t) = (1+t)^{-1}$) *and is independent of the total lifetime* $X+Y$ (*and similarly for* X/Y);

(d) *for* $0 \leq s \leq t$, $P(\min(X,Y) > s, \max(X,Y) > t) = 2S(s+t) - S(2t)$, *hence the probability that at least one of X and Y survives beyond time t is* $P(\max(X,Y) > t) = 2S(t) - S(2t)$, *and the probability that both X and Y survive beyond time t is* $P(\min(X,Y) > t) = S(2t)$;

(e) *the absolute difference* $|X-Y|$ *has the same distribution as X* (*and Y*), *i.e.*, $P(|X-Y| > t) = S(t)$;

(f) *the regression functions are* $E(Y|x) = -S(x)/S'(x)$ *and* $E(X|y) = -S(y)/S'(y)$.

Note that the regression function $E(Y|x) = -S(x)/S'(x)$ is the reciprocal of the *hazard function* (or failure rate) of X, given by $h(x) = -S'(x)/S(x)$, and similarly for $E(X|y)$.

Example 1 *Uniform distributions*. Let X and Y be uniformly distributed on the interval $[0,c]$ for some $c > 0$. Then $S(x) = \max(1 - x/c, 0)$, which is convex, and hence $S(x+y) = \max(1-(x+y)/c, 0)$ is a Schur-constant joint survival function. But since $S''(x+y) = 0$ a.e. on $[0,\infty)^2$, the joint distribution is singular, with the probability mass uniformly distributed on the line segment joining $(0,c)$ and $(c,0)$ in $[0,\infty)^2$. The (Archimedean) copula of X and Y, generated by $\varphi(t) = c(1-t)$, is $W(u,v) = \max(u+v-1, 0)$. W is the minimum copula, i.e., for any copula C, $C(u,v) \geq W(u,v)$ for u,v in [0,1].

Example 2 *Pareto and Weibull distributions.* (a) Suppose X and Y have a common Pareto distribution on $[0,\infty)$ with shape parameter θ, i.e., $S(x) = (1+x)^{-\theta}$, $\theta \in (0,\infty)$. Then $S(x+y) = (1+x+y)^{-\theta}$, a bivariate Pareto distribution first studied by

Mardia (1962). The copula of X and Y, generated by $\varphi(t) = t^{-1/\theta} - 1$, is $C(u,v) = \left(u^{-1/\theta} + v^{-1/\theta} - 1\right)^{-\theta}$, a member of the *Clayton* (1978) or *Cook and Johnson* (1981) family of Archimedean copulas. (b) Suppose X and Y have a common Weibull distribution on $[0,\infty)$ with shape parameter θ, i.e., $S(x) = \exp(-x^{\theta})$, $\theta \in (0,1]$; and thus $S(x+y) = \exp[-(x+y)^{\theta}]$. The copula of X and Y, generated by $\varphi(t) = (-\ln t)^{1/\theta}$, is $C(u,v) = \exp\left(-[(-\ln u)^{1/\theta} + (-\ln v)^{1/\theta}]^{\theta}\right)$, a member of the *Gumbel* (1960) or *Hougaard* (1986) family of Archimedean copulas.

3 Correlation, aging, and dependence properties

Let X and Y satisfy the hypotheses of Theorem 2. Since the joint distribution of (X,Y) depends only on the univariate survival function S, one might suspect that correlation coefficients and dependence properties for (X,Y) are related to parameters and aging properties of X (or Y) alone. This is indeed the case.

Since $P(X > x+y) = S(x+y) = P(X > x, Y > y)$, certain univariate aging properties of X (and Y) translate into dependence properties for the pair (X,Y). For example [Caramellino and Spizzichino (1994)], if X and Y satisfy the *new worse than used* (NWU) aging property , then $P(X > x) \le P(X > x+y | X > y)$ for all $x,y \ge 0$ (and similarly for Y), or equivalently $S(x+y) \ge S(x)S(y)$. But this means that $P(X > x, Y > y) \ge P(X > x)\, P(Y > y)$, i.e., X and Y are *positively quadrant dependent* (PQD), so that the probability that X and Y simultaneously have “long lives” is greater than it would be were X and Y independent. Similarly the new better than used (NBU) aging property is equivalent to the negatively quadrant dependent (NQD) dependence property. For another example, suppose that X and Y possess an *decreasing failure rate* (DFR), that is, $P(X > x+y | X > y)$ is nondecreasing in y for each x. This implies that $S(x+y)/S(y) = P(X > x | Y > y)$ is nondecreasing in y for each x, which is the *right tail increasing* (RTI) dependence property. Similarly, the increasing failure rate (IFR) aging property corresponds to the right tail decreasing (RTD) dependence property. For related results on aging properties of Schur-constant survival models, see Spizzichino (2001).

To evaluate Pearson’s product-moment correlation coefficient ρ, assume that Var(X), Var(Y), and $E(XY)$ exist and set $E(X) = E(Y) = \mu$ and Var(X) = Var(Y) = σ^2. A result of Hoeffding, e.g. Lehmann (1966), implies

$$E(XY) = \int_0^\infty \int_0^\infty S(x+y)\,dy\,dx = \int_0^\infty tS(t)\,dt = \frac{1}{2}E\left(X^2\right) = \frac{1}{2}\left(\sigma^2 + \mu^2\right),$$

and hence $\rho = \left[E(XY) - \mu^2\right]/\sigma^2 = (1/2)\left(1 - \mu^2/\sigma^2\right)$. Note that ρ is function of the coefficient of variation σ/μ for X, which measures the variability of X relative to its magnitude, and that $\rho > 0$ whenever $\sigma/\mu > 1$.

Since ρ fails to exist whenever Var(X) fails to exist, we also find a nonparametric measure of association between X and Y, the population version τ of Kendall's tau. Since the survival copula of X and Y is Archimedean with generator φ, e.g. Genest and MacKay (1986) and Joe (1997), we have

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}\,dt = 1 - 4\int_0^\infty x[S'(x)]^2\,dx\,.$$

In Table 1 we present several one-parameter families of lifetime distributions which possess convex survival functions (Pareto and Weibull from Example 2, as well as Gompertz, power, Beta, and Burr type XII) and consequently satisfy the properties in the preceding two sections. The table gives the univariate survival function $S_\theta(x)$ (the bivariate survival function is $S_\theta(x+y)$), the values of the parameter θ for which S_θ is convex, and the generator φ_θ of the Archimedean copula $C_\theta(u,v)$ $= S_\theta(\varphi_\theta(u)+\varphi_\theta(v))$.

Since X and Y are independent if and only if they are exponentially distributed, the table also gives values of Pearson's product-moment correlation coefficient and the population version of Kendall's tau.

Table 1 *Examples of lifetime distributions with convex survival functions.*

	$S_\theta(x)$	$\theta\in$	$\varphi_\theta(t)$	Pearson's ρ	Kendall's τ
Pareto	$(1+x)^{-\theta}$	$(0,\infty)$	$t^{-1/\theta}-1$	$\begin{cases}\text{d.n.e.} & \theta\le 2\\ 1/\theta & \theta>2\end{cases}$	$\dfrac{1}{1+2\theta}$
Weibull	$\exp(-x^\theta)$	$(0,1]$	$(-\ln t)^{1/\theta}$	(*)	$1-\theta$
Gompertz	$\exp(\theta(1-e^x))$	$[1,\infty)$	$\ln(1-(\ln t)/\theta)$	no closed form	$e^{2\theta}\,\mathrm{Ei}(-2\theta)^\dagger$
power	$\max(1-x^\theta,0)$	$(0,1]$	$(1-t)^{1/\theta}$	$1-\dfrac{1}{2}(1+\theta)^2$	$1-2\theta$
Beta (1,θ)	$[\max(1-x,0)]^\theta$	$[1,\infty)$	$1-t^{1/\theta}$	$-\dfrac{1}{\theta}$	$\dfrac{1}{1-2\theta}$
Burr type XII	$(1+x^{1/\theta})^{-\theta}$	$[1,\infty)$	$\left(t^{-1/\theta}-1\right)^\theta$	d.n.e.	$\dfrac{2\theta-1}{2\theta+1}$

with $^*\rho = 1-(1/2)\left[1-(1+(2/\theta))B(1+(1/\theta),1+(1/\theta))\right]^{-1}$, $^\dagger\mathrm{Ei}(x) = \int_{-\infty}^x e^t/t\;dt$.

4 Multivariate Schur-constant survival models

We now turn to the problem of characterizing multivariate Schur-constant survival models, that is, finding appropriate conditions on a univariate survival function S so that $S(x_1+\cdots+x_n)$ is a multivariate survival function for all $n\ge 2$. Note that $S(x_1+\cdots+x_n)$ is an n-dimensional survival function if and only if $S(\varphi(u_1)+\cdots+\varphi(u_n))$ is an n-dimensional survival copula. In Section 2 we saw that for $n = 2$, φ generates a copula when S is decreasing on $[0,\varphi(0)]$ and convex. Further conditions on S are required for $n > 2$, conditions which require the notion of a completely monotonic function.

Definition 1 [Widder (1941)]. A function $g(t)$ is *completely monotonic* on an interval I if it is continuous there and has derivatives of all orders which alternate in sign, i.e., if it satisfies

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0 \tag{4.1}$$

for all t in the interior of I and $k = 0,1,2,\cdots$.

If g is completely monotonic on $[0,\infty)$ and $g(c) = 0$ for some $c > 0$, then g must be identically zero on $[0,\infty)$ [Widder (1941)], hence if g is completely monotonic on $[0,\infty)$, then g is positive on $[0,\infty)$.

In the next theorem we show that completely monotonic survival functions can be used to construct n-dimensional Schur-constant survival models for any n.

Theorem 5 *Let S be a continuous strictly decreasing univariate survival function such that $S(0) = 1$. Then $S(x_1 + \cdots + x_n)$ is an n-dimensional survival function for all $n \geq 2$ if and only if S is completely monotonic on* $[0,\infty)$.

Proof. Since $S(x_1 + \cdots + x_n)$ can be written as $S(\varphi(S(x_1)) + \cdots + \varphi(S(x_n)))$ where φ is the inverse of S, the result follows from survival function versions of Theorems 1 and 2 in Kimberling (1974). ❑

If the joint survival function of the random vector $(X_1,\cdots,X_n)$ is $S(x_1 + \cdots + x_n)$, then $S(\varphi(u_1) + \cdots + \varphi(u_n))$ is an n-dimensional Archimedean copula with generator φ. Furthermore, if S is completely monotonic, then $S(x_1 + \cdots + x_n)$ is absolutely continuous for any $n \geq 2$.

The following theorem is a generalization to the multivariate case of several of the results in Theorem 4.

Theorem 6 *Let $\mathbf{X} = (X_1,\cdots,X_n)$ be a vector of n lifetimes with a common completely monotonic survival function S and a Schur-constant joint survival function $P(\mathbf{X} > \mathbf{x}) = S(x_1 + \cdots + x_n)$ for all $\mathbf{x} = (x_1,\cdots,x_n)$ in $[0,\infty)^n$. Then*

(a) *the survival function of the total lifetime (or "total time on test") $T = X_1 + \cdots + X_n$ is given by*

$$P(T > t) = \sum_{k=1}^{n-1} (-1)^k \frac{t^k}{k!} S^{(k)}(t) = (-1)^{n-1} \frac{t^n}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} \frac{S(t)}{t};$$

(b) *for $0 \leq s \leq t$,*

$$P(\min(\mathbf{X}) > s, \max(\mathbf{X}) > t) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} S(kt + (n-k)s),$$

and hence the probability that at least one X_i survives beyond time t is $P(\max(\mathbf{X}) > t) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} S(kt)$, and the probability that all X_i survive beyond time t is $P(\min(\mathbf{X}) > t) = S(nt)$;

(c) *if* $\{x_{i_1},\cdots,x_{i_m}\}$ *is a non-empty proper subset of the components of* $\mathbf{x}$, *and* $j \notin \{i_1,\cdots,i_m\}$, *then*

$$E\left(X_j \middle| x_{i_1},\cdots,x_{i_m}\right) = -S^{(m-1)}(x_{i_1}+\cdots+x_{i_m}) \big/ S^{(m)}(x_{i_1}+\cdots+x_{i_m}).$$

Proof. (a) Let C denote the Archimedean survival copula of $\mathbf{X}$ and φ its generator, and set $U_i = S(X_i)$ for $i = 1,\cdots,n$. Then

$$\begin{aligned} P(T > t) &= P(S(T) < S(t)), \\ &= P\big(C(S(X_1),\cdots,S(X_n)) < S(t)\big), \\ &= P\big(C(U_1,\cdots,U_n) < S(t)\big), \\ &= K_C(S(t)), \end{aligned}$$

where K_C denotes the distribution function of the random variable $C(U_1,\cdots,U_n)$. But

$$K_C(t) = t + \sum_{k=1}^{n-1} (-1)^k \frac{\varphi^k(t)}{k!} \frac{d^k}{ds^k} S(s)\Big|_{s=\varphi(t)}$$

[Barbe et al. (1996)], and thus

$$K_C(S(t)) = \sum_{k=0}^{n-1} (-1)^k \frac{t^k}{k!} S^{(k)}(t).$$

The second form for $P(T > t)$ is easily established by induction.

(b) Since $\min(\mathbf{X}) > s$ and $\max(\mathbf{X}) > t$ when all X_i are greater than s and at least one X_i is greater than t, the result is readily established by an inclusion-exclusion argument.

(c) This expression readily follows from the conditional probability

$$\begin{aligned} &P\left(X_j > x_j \middle| X_{i_1} = x_{i_1},\cdots,X_{i_m} = x_{i_m}\right) \\ &= -S^{(m)}(x_{i_1}+\cdots+x_{i_m}+x_j) \big/ S^{(m)}(x_{i_1}+\cdots+x_{i_m}). \end{aligned}$$ ❑

With the hypotheses of Theorem 6, we conjecture that the lifetime proportion $X_k/(X_1+\cdots+X_n)$, for k in $\{1,2,\cdots,n\}$, has a Beta distribution with parameters 1 and $n-1$, and is independent of $T = X_1+\cdots+X_n$.

Note added in proof: C. Genest (personal communication) has proven the above conjecture. Genest also notes that proofs of parts (a) and (b) of Theorem 6 appear in Genest and Rivest (1993).

There is an intermediate case between the situations discussed in Section 2 (S decreasing and concave) and in this section (S completely monotonic). When (4.1) holds on $(0,\varphi(0))$ for $k = 0,1,\cdots,m$, we say that S is *m-monontonic*. The arguments in Kimberling (1974) can be used to show that the m-monotonicity of S is sufficient for $S(x_1+\cdots+x_n)$ to be a n-dimensional survival function for $2 \le n \le m$.

Example 3 *Beta* $(1,\theta)$ *distributions*. Let S be the survival function for a Beta$(1,\theta)$ random variable, i.e., $S(x) = [\max(1-x,0)]^\theta$, for $\theta \ge 1$ (see Table 1). Then S is m-

monotonic for $m = \lfloor\theta\rfloor+1$, and hence $S(x_1+\cdots+x_n) = [\max(1-x_1-\cdots-x_n,0)]^\theta$ is an n-dimensional Schur-constant survival function for $2 \le n \le \lfloor\theta\rfloor+1$. This is the Schur-constant subfamily of the multivariate Dirichlet distribution in Kotz et *al.* (2000).

5 Concluding remarks

Closely related to Schur-constant functions are Schur-convex and Schur-concave functions. If (x_1,y_1) and (x_2,y_2) are two points in $\mathbf{R}^2$, then (x_1,y_1) is *majorized* by (x_2,y_2), written $(x_1,y_1)\prec(x_2,y_2)$, if $x_1+y_1 = x_2+y_2$ and $\max\{x_1,y_1\} \le \max\{x_2,y_2\}$ (majorization is defined in higher dimensions as well, see Marshall and Olkin (1979), Spizzichino (2001) for details). A function $f{:}A\subseteq\mathbf{R}^2\to\mathbf{R}$ is *Schur-convex* (*-concave*) *on A* if $(x_1,y_1)\prec(x_2,y_2)$ implies $f(x_1,y_1) \le (\ge)\ f(x_2,y_2)$ for any (x_1,y_1), (x_2,y_2) in *A*. Functions which are both Schur-convex and -concave are Schur-constant.

Recently, Durante and Sempi (2003) showed that all Archimedean copulas (and many non-Archimedean copulas as well) are Schur-concave, but only the minimum copula *W* (see Example 2.1) is Schur-convex (and since *W* is Archimedean, it is thus the only Schur-constant copula). However, when endowed with marginal survival functions, Schur-concave Archimedean copulas may yield Schur-concave, -convex, or -constant joint survival functions. For example, if the Gumbel-Hougaard copula $\exp\left(-[(-\ln u)^{1/\theta}+(-\ln v)^{1/\theta}]^\theta\right)$ is endowed with Weibull marginal survival functions $S(x) = \exp(-x^\alpha)$ and $S(y) = \exp(-y^\alpha)$, then the resulting joint survival function is $\overline{F}(x,y) = \exp\left(-\left[x^{\alpha/\theta}+y^{\alpha/\theta}\right]^\theta\right)$. Since the level curves of $\overline{F}$ are given by $x^{\alpha/\theta}+y^{\alpha/\theta}$ = constant, $\overline{F}$ is Schur-concave for $\alpha \ge \theta$ and Schur-convex for $\alpha \le \theta$.

Since, as noted in the remark preceding Example 1, there is a one-to-one correspondence between Archimedean copulas and scale-parameter families of Schur-constant survival models, Schur properties may play a role in answering the survival copula version of the following open problem, e.g., Alsina et *al.* (2003): "There are numerous statistical arguments that are used to justify the assumption of normality. Are there similar arguments that can be used to justify the assumption that the copula of two random variables is Archimedean?"

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Roger B. Nelsen
Department of Mathematical Sciences
Lewis & Clark College
Portland, OR 97219, USA
E-mail: nelsen@lclark.edu